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Resonance Control of Aircraft Instabilities by Smooth and Continuous Feedback

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Analysis and control of bifurcation phenomena occurring in high angle-of-attack aircraft flight have been the subject of a number of recent studies. It has been shown that smooth nonlinear feedback is capable of stabilizing nonlinear transient phenomena in a certain range of parameters of aircraft models. In a recent paper we presented an integrated approach employing the normal forms reduction methodology for both analysis and control of aircraft instabilities. In this paper we describe a novel resonance control design approach that assumes feedback as a composition of continuous (nonsmooth) and smooth nonlinear components. We show that such compound resonance control matches superior performance with a reduction of feedback magnitude required for stabilization.

Introduction

OPERATING near the extrema of the flight envelope often enhances maneuverability of aircraft. In fact, small perturbations of such regimes may cause unstable behavior, especially in cases of high angle-of-attack flights. Observed instabilities of aircraft have been linked with bifurcations of steady-state regimes of aircraft models in a number of recent publications.[1–4] Attention has also been given recently to the study of control design in critical nonlinear systems faced with bifurcations (see Ref. 5 and references therein). In Refs. 6 and 7 it was shown that certain simple aircraft instabilities can be stabilized by appropriate feedback control. A novel integrated approach providing analysis and resonance control of aircraft instabilities was addressed in our recent paper,[8] where a resonance feedback is designed as a nonlinear function of deviations from the normal operating regime. Design of resonance control is based on the normal forms reduction methodology, which intrinsically voids the nonresonance portion of both a system and control that is insignificant in stabilization (see Ref. 9 for further details about the normal forms reduction methodology). In this paper we address the design of a more efficient compound resonance control structured with continuous (nonsmooth) and conventional smooth nonlinear components. A continuous component is formed to reduce small perturbations, whereas the smooth one is designed to suppress larger distortions. Such compound control adopts profound features of smooth and discontinuous design approaches and escapes their side effects. We show that compound control reduces high feedback magnitude required for stabilization and ensures fast decays of perturbations in a wide region of system parameters.

The remainder of the paper is organized as follows. In the next section we observe briefly a close relation between bifurcation and stability transient phenomena and review the Poincaré normal forms method. Then we address a more general and computationally efficient reduction approach, termed a generalized normal forms method, which is structured as successive approximations. The residual of these truncated approximations is bounded in the following section. In the subsequent section a method of designing a resonance control is described. After that we compare efficiency of smooth and compound resonance control in the stabilization of the Hopf bifurcation. The next section describes analysis and resonance control of instabilities of longitudinal dynamics of aircraft linked with the saddle-node bifurcation. The final section concludes our studies.

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Normal Forms Method and Bifurcation Analysis

Robust feedback stabilization of a normal flight regime is one of the principal objectives in aircraft guidance. Because stability analysis and stability control are closely related, we begin with a brief observation of a profound methodology that significantly simplifies a local analysis of nonlinear transient behavior (bifurcations) of a normal operating regime occurring due to a change in system parameters. Next, we engage a stabilizing feedback control as a function of deviations from this operating regime, which ensures asymptotic decay of these deviations.

Aircraft models governing local transition behavior of a normal operating regime of an aircraft can be written in the following form:

$$\dot{x} = A(\delta)x + \varepsilon f(x, \delta, \varepsilon) \tag{1}$$

where $x \in R^n$, f is a power series in x, and $f(0, \delta, \varepsilon) = 0$, A is a square diagonal matrix of order n, diag $A = \{\lambda_1, \dots, \lambda_n\}$, $\delta \in R^m$ is a parameter vector, and $0 < \varepsilon$ is a small parameter linked with the neighborhood size around $x = 0$. Local stability analysis describes stability transition of a static solution $x = 0$ that occurs when eigenvalues of A cross the imaginary axis due to alteration of δ. Poincaré has drawn attention to the fact that analysis of nonlinear stability transition of the trivial solution $x = 0$ of Eq. (1) is coupled with revealing the corresponding bifurcation phenomenon of a branching of the static solution. Indeed, the global bifurcation analysis is rather complex and has been completed only in relatively simple cases. Fortunately, just local analysis of bifurcation phenomena often provides information essential in engineering practice.

As is known,[9] small nonlinear terms affect a local transient behavior of Eq. (1) because the larger linear terms are partially annihilated near the bifurcation points. Recall that linearized stability analysis is clearly expressed in terms of eigenvalues of A, which represent invariants of the linear coordinate transformations. Description of peculiarities of local nonlinear stability transition is also greatly simplified when expressed in terms of the invariants of local nonlinear changes of variables provided by the normal forms reduction technique. In fact, nonlinear normal forms play the same principal role in revealing local nonlinear stability transient behavior as normal forms of matrices play in the linear case.

Let us briefly describe the Poincaré normal forms method that attempts to simplify Eq. (1) with the aid of a close-to-identity change of variables

$$x = y + \varepsilon r(y, \varepsilon) \tag{2}$$

Assume now that $f = f_{ms} = x^m e_s$, where $x^m = x_1^{m_1} \dots x_n^{m_n}$, $m_i, i = 1, \dots, n$, are positive integers, $m_1 + \dots + m_n \geq 2$, $f_{ms} = \text{const}$, and e_s is a unit vector of the s axis (namely, f is a vector having a single sth nonzero component). Then Eq. (2)

annihilates f in Eq. (1) if r satisfies the so-called homological equation

$$L_A r = f(x) \tag{3}$$

where $L_A r = (\partial r/\partial x)Ax - Ar$ is a Lie bracket and $\partial r/\partial x$ is the Jacobian of r. For the chosen form of f, Eq. (3) has a particular solution $r = r_{ms}x^m e_s$, where $r_{ms} = f_{ms}/[(m; \lambda) - \lambda_s]$, $(m; \lambda) = m_1\lambda_1 + \cdots + m_n\lambda_n$, $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is a vector formed from the eigenvalues of A. In fact, monomials conforming with the Poincaré resonance condition

$$\lambda_s = (m; \lambda) \tag{4}$$

persist in Eq. (1) and are termed the resonance monomials. According to the Poincaré method, all nonresonance monomials have to be eliminated, beginning with monomials of the lowest degree, to obtain the normal form. The normal form consists only of resonant monomials and turns out to be a significantly reduced equation.

Early reduction techniques used in stability analysis, such as Malkin's method,[10] yield a first-order approximation to a center manifold reduction in the cases of a single zero or single complex conjugate pair of critical eigenvalues. The normal forms methodology provides successive approximations to a center manifold in these and more complex cases and also yields normalization of the equations on the center manifold. This reduction plays a significant role in local bifurcation analysis. The critical eigenvectors (coupled with the eigenvalues located on the imaginary axis) span a hyperplane that is tangent to the center manifold at the equilibrium point. The local approximation provided by Eq. (2) successively decouples the equations on the center manifold from the remaining equations. A remarkable theorem in bifurcation theory asserts that the behavior of a local bifurcation can be completely revealed on the center manifold if it is a sufficiently smooth vector function.

Let us consider how the normal forms reduction simplifies the bifurcation analysis in the two simple cases of concern in bifurcation theory. First assume that a single critical eigenvalue $\lambda_s = 0$. From Eq. (4) it follows that the center manifold in this case is single directional and consists of the following normalized equations that include the infinite series of resonance monomials:

$$\dot{x}_s = \sum_{m=2,\ldots,} q_{ms} x_s^m e_s$$

where q_{ms} are obtained through the normalization procedure. The remaining normalized equations are decoupled from the preceding and can be disregarded in bifurcation analysis. Next, assume that A possesses a pair of pure imaginary (critical) eigenvalues (λ_s and $\bar{\lambda}_s$) and the remaining (noncritical) eigenvalues are located off of the imaginary axis. In this case Eq. (4) together with the identity $\lambda_s + \bar{\lambda}_s = 0$ leads to the following resonance condition: $\lambda_s = (n+1)\lambda_s + n\bar{\lambda}_s$, $n = 1, 2, \ldots,$ and the normal forms equations projected on a two-dimensional center manifold also consist of an infinite series of resonance monomials:

$$\begin{aligned} \dot{x}_s &= x_s\left[\lambda_s + \sum_{m\ge 1} q_{ms}(x_s\bar{x}_s)^m\right] \\ \dot{\bar{x}}_s &= \bar{x}_s\left[\bar{\lambda}_s + \sum_{m\ge 1} \bar{q}_{ms}(x_s\bar{x}_s)^m\right] \end{aligned} \tag{5}$$

where complex conjugate q_{ms} and $\bar{q}_{ms}$ are given by the normalization algorithm. We notice that the remaining normalized equations are decoupled from Eq. (5) and are insignificant in bifurcation analysis.

A bifurcation phenomenon is considered in a parametric space allowing a bifurcation to arise generically. The bifurcation theory specifies a truncation of the normal forms infinite series preserving the local bifurcation characteristics of Eq. (1) (see Ref. 9). The truncated normal form specifies invariants that capture peculiarities of a local stability transition of a static solution of Eq. (1). Notice that, as a rule, a neighborhood of validity of the Poincaré method is not estimated rigorously, which complicates utility of the standard normal forms method in applied stability analysis.

The normal forms theory provides an attractive opportunity for stabilizing nonlinear transient behavior by feedback designed as a function of the resonance terms with adjustable coefficients. Such a resonance feedback will intrinsically void the bulk nonresonance components of a system insignificant in stabilization and appears to be an efficient and simple control design approach. Notice that robust stabilization of a nonlinear plant by a linear feedback often increases control magnitude, especially if stability is to be ensured in an extended neighborhood about the operating regime. Such overcontrol degrades aircraft maneuverability and may cause other side effects as well. In the next three sections we describe a unified methodology providing the normal forms reduction and design of nonlinear resonance control of bifurcation phenomena occurring in systems modeling aircraft flight.

Generalized Normal Forms Method

Unfortunately, lack of computability complicates adoption of the standard normal forms method in solving many applied problems. We describe here a more general approach to finding normal forms that is more efficient in computations and is applicable to a broader class of systems than the known techniques.

It is remarkable that in the context of the following algorithm the nonlinear resonance condition is shaped in a form congruent with an analogous relation from the theory of linear forced oscillations. Although it is easy to verify that the Poincaré resonance monomials are captured by the new condition, it is important to note that this approach may also be used when the known relation is invalid. Note also that for equations written in the normal form, separation of slow and fast variables becomes rather routine and is presented in a closed form in this section.

Suppose that in the new variables Eq. (1) takes the form

$$\dot{y} = Ay + \varepsilon B(y, t, \varepsilon) \tag{6}$$

From Eqs. (1), (2), and (6) we get

$$Ar - r_y Ay - r_t = -f(y + \varepsilon r) + B + \varepsilon r_y B \tag{7}$$

where r_y is the Jacobian with respect to y and r_t is a partial derivative of r with respect to t. We consider Eq. (7) as a system of partial differential equations with respect to r. We seek a particular solution of Eq. (7) satisfying an additional condition:

$$\|f\| \to 0 \Rightarrow \|r\| \to 0$$

where $\|\cdot\|$ is a certain norm of a vector valued function. In other words, we are looking for a particular solution such that the norm of r tends to zero if the norm of f tends to zero also. For small ε one can try to approximate the solution of Eq. (7) by the following successive approximations:

$$Ar_1 - r_{1,y}Ay - r_{1,t} = -f(y)$$

$$\dot{r}_k - r_{k,y}Ay - r_{k,t} = -f(y + \varepsilon r_{k-1}) + B_k + \varepsilon r_{k-1,y}B_{k-1}, \qquad k > 1 \tag{8}$$

Note that Eq. (8) represents a set of linear partial differential equations with respect to r_k, $k > 1$. Let us write a characteristic equation for Eq. (8):

$$\dot{r}_1 = Ar_k + f(y, t) - B_1 \tag{9}$$

$$\dot{r}_k = Ar_k + f(y + \varepsilon r_{k-1}) - \varepsilon r_{k-1,y}B_{k-1} - B_k, \qquad k > 1 \tag{10}$$

$$\dot{y} = Ay \tag{11}$$

From Eq. (11) we see that $y = e^{At}c$, where $c = \{c_1, \ldots, c_n\}$ is a constant vector.

The time-dependent terms in Eqs. (9) and (10) should be split up into two distinct groups, namely, resonance and nonresonance terms.

Definition. The term $F(t)$ is called a resonance function if it can be represented in the following form:

$$F(t) = e^{At}N(c) \tag{12}$$

where $N(c) \in R^n$ is a vector dependent upon a constant vector c; otherwise F is called a nonresonance function. To annihilate nonresonance terms we set $B_k = e^{At}N_k(c)$ (recall that $c = e^{-At}y$). So one is able to write

$$B_k(y, t) = e^{At}N_k(e^{-At}y) \tag{13}$$

Equations (12) and (13) naturally express a nonlinear resonance condition in a form adopted from the theory of linear forced oscillations.

Although Eq. (13) looks distinct from Eq. (6), it turns out to be identical to the Poincaré condition in the case when $f_s(x) = f_{ms}x^m e_s$, f_{ms} = const. Recalling that A is a diagonal matrix, and $f_s(e^{At}c) = f_{ms}e^{(m;\lambda)t}c^m e_s$, we see that f_s admits form (11) if $(m; \lambda) = \lambda_s$. Note also that in this case $B_s = \alpha_{ms}y^m e_s$.

Observe that Eqs. (9–11) coincide with the set of linear ordinary differential equations (ODEs) given by simple successive approximations if one sets $B_k = 0, k > 1$. As is known, the resonance terms usually present in these recursive sequences force these iterations to diverge on large time intervals. The problem with such successive approximations is that they do not distinguish between resonance and nonresonance terms. In spite of this, one can use the available B_k to annihilate resonance terms in Eqs. (9) and (10) and place them into the normal form. Because the resonance terms are exceptional, the normalization yields significant reduction of the initial system.

The normal forms admit a closed-form complementary reduction in both the so-called slow and especially in the amplitude-phase variables, which inherently occur in the presence of complex-conjugate eigenvalues. The imaginary parts of the eigenvalues correspond with the oscillatory components of solutions that are naturally filtered out when the normal forms equations are written in the amplitude-phase variables.

Let us write the normal form equations as follows:

$$\dot{y} = Ay + \varepsilon e^{At}N(e^{-At}y, \varepsilon) + \mathcal{O}(\varepsilon^k) \tag{14}$$

Denote a slow variable $c(t)$ by the formula

$$y(t) = e^{At}c(t)$$

Then one obtains

$$\dot{c} = \varepsilon N(c, \varepsilon) + \mathcal{O}(\varepsilon^k) \tag{15}$$

As is known, a couple of complex-conjugate eigenvalues in A (say λ_k and $\bar{\lambda}_k$) relate to a couple of complex-conjugate functions c_k and $\bar{c}_k$ in Eq. (15). Let us show that in such cases we are able to rewrite the last equation for complex functions as real differential equations.

Assuming now for simplicity that all eigenvalues are complex-conjugate couples:

$$\lambda_k = \alpha_k + I\omega_k, \qquad \bar{\lambda}_k = \alpha_k - I\omega_k$$
$$k = 1, \dots, n/2, \qquad I = \sqrt{-1}$$

we present related c_k and $\bar{c}_k$ in the following exponential form:

$$c_k(t) = a_k(t)e^{I\rho_k(t)}, \qquad \bar{c}_k(t) = a_k(t)e^{-I\rho_k(t)}$$
$$k = 1, \dots, n/2 \tag{16}$$

where a_k and ρ_k are real functions, known as the amplitude and phase variables. Let us consider the application of (a, ρ) variables to the normal forms equation in the case when all eigenvalues of A are located close to the imaginary axis [i.e., that $\alpha_k = \mathcal{O}(\varepsilon)$, $k = 1, \dots, n/2$] that is the main concern in local bifurcation theory relating a bifurcation phenomenon with the transition of eigenvalues through the imaginary axis. With that assumption, we will write $\varepsilon\alpha_k$ instead of α_k, where now $\alpha_k = \mathcal{O}(1)$. Next, we present Eq. (1) in a modified form:

$$\dot{x} = (\text{Im}\, A)x + \varepsilon F(x, \delta, \varepsilon) \tag{17}$$

where matrix Im $A = \text{diag}[I\omega_1, -I\omega_1, \dots, I\omega_{n/2}, -I\omega_{n/2}]$, Re $A = \text{diag}[\varepsilon\alpha_1, \varepsilon\alpha_1, \dots, \varepsilon\alpha_{n/2}, \varepsilon\alpha_{n/2}]$, and $\varepsilon F = \varepsilon f + (\text{Re}\, A)x$. Now we observe that the eigenvalues of the matrix Im A admit natural resonance conditions: $\omega_k + (-\omega_k) = 0$, $k = 1, \dots, n/2$. We assume for simplicity that there are no other resonance conditions or that there is no vector m [as in Eq. (4)] that admits the Poincaré condition complementary to the natural ones. The application of the generalized normal forms method to Eq. (17) yields the normal forms equations that can be presented in the amplitude-phase variables (16) in the following form:

$$\begin{aligned} \dot{a} &= \varepsilon\alpha a + \varepsilon\text{Re}[N(a, \varepsilon)] + \mathcal{O}(\varepsilon^k) \\ \dot{\rho} &= -\varepsilon a^{-1}\text{Im}[N(a, \varepsilon)] + \mathcal{O}(\varepsilon^k) \end{aligned} \tag{18}$$

where $a = \{a_1, \dots, a_{n/2}\}$, $\rho = \{\rho_1, \dots, \rho_{n/2}\}$, and $\mathcal{O}(\varepsilon^k)$ stands for terms of order ε^k that have been disregarded. The most notable feature of the preceding equations is that the amplitude variables are decoupled from the phase ones.

In the complementary case when all real $\lambda_k = \mathcal{O}(\varepsilon)$, we may write Eq. (1) in the form

$$\dot{x} = \varepsilon F(x, \delta, \varepsilon)$$

where $\varepsilon F = \varepsilon f + Ax$. Applied to that equation, the generalized normalization technique yields the real normal form equations that can be written as Eq. (15).

Remark 1. In practice, such adjusting of the eigenvalues (which we term critical or close to critical) should be done when they cross or closely approach the imaginary axis due to δ variation. The normalization decouples the critical equations related to critical or close-to-critical eigenvalues from the remaining ones, which in the majority of applied problems turn out to be stable. Ignoring these circumstances yields a normal form that is not capable of capturing a local bifurcation behavior resulting from transition of certain eigenvalues through the imaginary axis due to variation of δ.

Notice that when critical eigenvalues consist of both a number of zero eigenvalues and a certain number of noncommensurable critical complex-conjugate couples, the normal form in (a, ρ) variables can be written as Eq. (18) if we formally identify the slow variables related to zero eigenvalues with additional components in the vector a.

In the cases when the resonances complementary to natural ones $[\omega_k + (-\omega_k)]$ occur, $\omega_s = (m; \omega)$, where $s = 1, \dots, p$ and the positive integer vector m includes at least one component, say $m_l \neq 0$, $l \neq s, s+1$ and m may depend upon s, conforming combinations of phase variables, namely,

$$\psi_s = (m; \rho) - \rho_s, \qquad s = 1, \dots, p \tag{19}$$

persist in Eq. (18). Defining by Eq. (19) the new variables $\psi = \{\psi_1, \dots, \psi_p\}$ and leaving the same notation for the remaining variables, we write Eq. (18) in the modified form

$$\begin{aligned} \dot{a} &= \varepsilon\alpha a + \varepsilon\text{Re}[N(a, \psi, \varepsilon)] + \mathcal{O}(\varepsilon^k) \\ \dot{\rho} &= -a^{-1}\varepsilon\text{Im}[N(a, \psi, \varepsilon)] + \mathcal{O}(\varepsilon^k) \\ \dot{\psi} &= -a^{-1}\varepsilon\text{Im}[N(a, \psi, \varepsilon)] + \mathcal{O}(\varepsilon^k) \end{aligned} \tag{20}$$

where $k > 1$. Thus additional p resonances yield additional p active variables: ψ in the averaging normal form. Note that the last equation represents the effect of $(k-1)$ iterations that yield averaging in the (a, ρ) variables as well.

Estimation of Residual Terms

As is known (see Ref. 9 for further details), successive normalization often diverges in practically important cases. Indeed, truncated normalization could be a valuable source in bifurcation analysis if the residuals are properly estimated. Equating

$$B_k = f(y + \varepsilon r_{k-1}) - \varepsilon r_{k-1,y}B_{k-1}, \qquad k > 1$$

we get $r_{k-1} = r_k$, which means that successive normalization has been stopped on the $k-1$ step. In this case, B_k consists of the resonance and nonresonance terms as well. Accumulating the nonresonance terms in residuals, we write the truncated normal forms equations in the amplitude-phase variables as follows:

$$\dot{a} = \varepsilon \alpha a + \varepsilon \mathrm{Re}[N(a, \varepsilon)] + \varepsilon^k E(a, \rho, \varepsilon)$$
$$\dot{\rho} = -a^{-1}\varepsilon \mathrm{Im}[N(a, \varepsilon)] + \varepsilon^k e(a, \rho, \varepsilon) \tag{21}$$

where amplitudes $a \in R^{n_1}$ and phases $\rho \in R^{n_2}$ ($n_1 - n_2$ is the number of zero eigenvalues in matrix α), and residuals of truncated normalization E and e are periodic functions with respect to each ρ_k with noncommensurable periods T_k:

$$E(a, \rho_k + T_k) = E(a, \rho_k), \qquad e(a, \rho_k + T_k) = e(a, \rho_k)$$

Note that Eqs. (21) are accurate equations derived from Eq. (1) by the closed-form changes of variables. In fact, residuals depend upon both a and ρ variables, which forces us to use in stability analysis their ρ-invariant upper bounds, which we write as follows:

$$E \le \max_{\rho_1 \in [0;T_1]} \cdots \max_{\rho_n \in [0;T_n]} E(a, \rho, \varepsilon) = E^0(a, \varepsilon)$$
$$e \le \max_{\rho_1 \in [0;T_1]} \cdots \max_{\rho_n \in [0;T_n]} e(a, \rho, \varepsilon) = e^0(a, \varepsilon) \tag{22}$$

Using these bounds we can formulate asymptotic stability conditions for a steady-state solution of Eq. (21) with the aid of a Lyapunov function addressed in amplitude variables. Indeed, there are valuable advantages in application of the Lyapunov approach to the averaged equations (21) rather than to the initial ones. The averaged equations govern the evolution of slow variables that envelop trajectories of the initial system and select the information valuable for stability analysis from the routine ones. Thus, relatively simple Lyapunov functions defined in the amplitude variables provide sharp sufficient stability conditions that in many cases assume explicit physical interpretation. It is unclear how to select such a Lyapunov function directly in the original variables. In addition, the normal forms represent the irreducible model of an initial plant that discloses the mechanism of a plant's stability transition. These models admit simplified numerical analysis and often closed-form integration.

Resonance Feedback Stabilization

Assume now that control $u = \{u_1, \ldots, u_n\}$ enters Eq. (1) in the following way:

$$\dot{x} = A(\delta)x + \varepsilon f(x, \delta, \varepsilon) + u(x, \delta, \varepsilon) \tag{23a}$$

where u is a polynomial vector function and $u(0, \delta, \varepsilon) = 0$. In this case u describes a portion of control stabilizing the $x = 0$ solution of Eq. (23a) that exhibits transients when δ reaches the bifurcation value $\delta = \delta^0$. A complimentary portion of control (which may alter the location of a steady-state solution) is assumed to be adjoint to f. Direct application of a Lyapunov-based approach to Eq. (23a) yields a design for u that ignores the distinction between resonance and nonresonance terms, leading to overcontrol. The resonance feedback stabilization method takes advantage of the normal forms methodology by reducing controlled plant (23a) (with $u = 0$) to the normal form (21). Next, Eq. (21) is stabilized by control $U(a, \delta)$ designed as a function of the amplitude variables. Notice that the effect of bounded unmodeled dynamics on the resonance feedback stabilization may be accounted for by formally collecting these bounds that are mapped into the amplitude-phase variables into the residual bounds. Finally, feedback $U(a, \delta)$ is mapped into the original variables x.

To explain the design of resonance feedback control, we write a controlled plant's equations in the amplitude-phase variables as follows:

$$\dot{a} = \varepsilon G(a, \rho, \delta, \varepsilon)$$
$$\dot{\rho} = \varepsilon g(a, \rho, \delta, \varepsilon) \tag{23b}$$

where $G(a, \rho, \delta, \varepsilon) = Q(a, \delta, \varepsilon) + \varepsilon E(a, \rho, \delta, \varepsilon) + \varepsilon U(a, \delta)$, $g(a, \rho, \delta, \varepsilon) = q(a, \delta, \varepsilon) + \varepsilon e(a, \rho, \delta, \varepsilon)$, Q and q are resonance terms, $Q(0, \delta, \varepsilon) = q(0, \delta, \varepsilon) = 0$, E and e are residuals, $E(0, \rho, \delta, \varepsilon) = e(0, \rho, \delta, \varepsilon) = 0$, the control $U(a, \delta)$ has the property $U(0, \delta) = 0$. It is clear that we are controlling only the amplitude variables since Eq. (16) guarantees that explicit evolution of the phase variables is insignificant in stability analysis. However, their implicit effect on the amplitude variables is accounted for by the bound of E. Notice also that Eq. (23a) is mapped to form Eq. (23b) by a closed-form change of variables obtained by truncating a successive normalization (2). Thus, a stabilization of an $x = 0$ solution of Eq. (23a) results from stabilization of a relevant solution $a = 0$ of Eq. (23b).

Now using Eq. (22), we write

$$|G(a, \rho, \delta, \varepsilon)| < G^0(a, \delta, \varepsilon)$$

where $G^0 = Q(a, \delta, \varepsilon) + E^0(a, \delta, \varepsilon) + \varepsilon U(a, \delta)$. Next, we engage the Lyapunov stability condition with the aid of a Lyapunov function $L = L(a)$ defined in a neighborhood around the origin $a = 0$:

$$\left.\frac{\mathrm{d}L(a)}{\mathrm{d}t}\right|_{\dot{a}=\varepsilon G} \le \sum_{i=1}^{n} \frac{\partial L}{\partial a_i} G_i^0(a, \delta, \varepsilon) \le -d$$
$$\mu \le |a| \le a^0, \qquad \mu > 0 \tag{24}$$
$$\left.\frac{\mathrm{d}L(a)}{\mathrm{d}t}\right|_{\dot{a}=\varepsilon G} \le \sum_{i=1}^{n} \frac{\partial L}{\partial a_i} G_i^0(a, \delta, \varepsilon) \le 0, \qquad |a| \le \mu$$

where vector $d > 0$ is assigned the degree of relative stability, vector a^0 initializes a basin of stability, and μ is a small positive value. Note that, in the last relation, equality is achieved only for $a = 0$.

Although a variety of Lyapunov functions can be adapted in feedback design, the most simple one $L = \sum_{i=1}^{n_1} a_i^2$ often yields a desirable controller. Using the fact that $a_i \ge 0$ when all critical/close-to-critical eigenvalues are complex conjugate, we write Eq. (24) in the reduced form

$$G_i^0(a, \delta, \varepsilon) \le -d, \qquad \mu \le a \le a^0$$
$$G_i^0(a, \delta, \varepsilon) \le 0, \qquad 0 \le a < \mu \tag{25}$$

The last inequalities can be written in the form

$$\dot{a}_i \le -d_i, \qquad \mu \le a \le a^0$$
$$\dot{a}_i \le 0, \qquad 0 \le a < \mu$$

disclosing that the resonance feedback controls least rates d_i of exponential decay of the amplitude variables that is a principal concern in various engineering applications.

Let all critical eigenvalues be real numbers close or equal to zero. In this case a_i may take positive or negative values. In particular, for a single zero eigenvalue Eq. (24) reduces to

$$aG^0(a, \delta, \varepsilon) \le -d, \qquad \mu \le |a| \le a^0$$
$$aG^0(a, \delta, \varepsilon) \le 0, \qquad |a| < \mu$$

where G^0 conforms with the last relation if it admits the property

$$\mathrm{sign}\, G^0(a, \delta, \varepsilon) = -\mathrm{sign}\, a$$

The last relation is satisfied if G^0 is chosen, for example, as an appropriate smooth odd function.

In the case of multiple zero/close-to-zero eigenvalues, Eq. (24) is reduced to

$$\sum_{i=1}^{n_1-n_2} a_i G_i^0(a, \delta, \varepsilon) \le -d, \qquad \mu \le |a| \le a^0$$
$$\sum_{i=1}^{n_1-n_2} a_i G_i^0(a, \delta, \varepsilon) \le 0, \qquad |a| < \mu \tag{26}$$

These inequalities agree if

$$\mathrm{sign}\, G_i^0(a, \delta, \varepsilon) = -\mathrm{sign}\, a_i, \qquad G_i^0(a = 0, \delta, \varepsilon) = 0$$
$$\left|G_i^0(a, \delta, \varepsilon)\right| < d_i, \qquad \mu \le |a| \le a^0$$

where rates of decay d_i are determined from Eq. (26).

The general case when both complex and real critical eigenvalues persist can be broken out into the two special cases mentioned earlier.

Remark 2. As has already been mentioned, in the majority of engineering problems the stability transition is captured by low-dimensional center manifold equations that are decoupled from the remaining (as a rule) stable equations. Thus, stable off-center manifold equations can be left uncontrolled by the resonance feedback. It is clear that the utility of that as well as other properties of the averaged equations significantly simplifies and improves the efficiency of the resonance control design.

The given stability conditions do not shape the procedure of control design uniquely. We examine with the following example one of the possible ways to choose smooth and continuous resonance stabilizing feedbacks.

Smooth and Continuous Resonance Control of the Hopf Bifurcation

Now we apply both the normal forms analysis and the resonance feedback stabilization to controlling instabilities in the Van der Pol equations

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1(x_1, x_2) \\ \dot{x}_2 &= -x_1 + \varepsilon\left(x_2 - x_2^3\right) + u_2(x_1, x_2) \end{aligned} \tag{27}$$

where $u = \{u_1, u_2\}$ is a control and $u_i(0,0) = 0$, $i = 1, 2$. Consider first a transient behavior occurring in the uncontrolled equations ($u = 0$) that are known as a simple model revealing the Hopf bifurcation. The eigenvalues of the unperturbed system (27) (with $\varepsilon = 0$) are $\omega_1 = I$, $\omega_2 = -I$. These eigenvalues satisfy the Poincaré resonance condition that follows from the identity, $\omega_1 + \omega_2 = 0$, and the normal form takes the form of Eq. (5).

Assuming $\varepsilon = 0.1$, we deduce the first-order approximation to the normal form for Eq. (27) with $u = 0$ and write these equations in (a, ρ) coordinates as

$$\begin{aligned} \dot{a} &= 0.05a - 0.15a^3 + \varepsilon^2 E(a, \rho) \\ \dot{\rho} &= \varepsilon^2 e(a, \rho) \end{aligned} \tag{28}$$

where residuals $E(a, \rho)$ and $e(a, \rho)$ are obtained in the second iteration as previously mentioned.

The trivial solution $a = 0$ is unstable in this case, but the system has a stable limit circle. Now we are able to engage feedback $U(a)$ and the residual bound

$$E(a, \rho) < E^0 = 0.00928a^3 + 0.0346a^5 + 0.00253a^7 + 0.0002a^9$$

in the control design. Combining the resonance, residual, and feedback terms, we get

$$G_i^0(a) = 0.05a - 0.141a^3 + 0.0346a^5 + 0.00253a^7 + 0.0002a^9 + U_i(a)$$

Because in this case $a \geq 0$, the positive terms in G_i^0 contribute to instability [see Eq. (25)]. There are various ways to choose $U_i(a)$ to satisfy Eq. (25). We choose a smooth feedback

$$U(a) = ka - 0.0350a^5 - 0.00260a^7 - 0.000210a^9 \tag{29}$$

(where k is an adjustable parameter) that annihilates the small destabilizing (positive) nonlinear terms in G_i^0 and preserves the main nonlinear term ($-0.141a^3$) that contributes to stability

Let $\mu = 0.05$ and $d = 0.1$; then determining k that satisfies Eq. (25), we get

$$U(a) = U_1(a) = -2.05a - 0.0350a^5 - 0.00260a^7 - 0.000210a^9 \tag{30}$$

Mapping this function back to the original variables introduces control in measurable initial coordinates as follows:

$$u_{11}(x_1, x_2) = -2.05x_1 - 0.0022x_1^5 - 0.0044x_1^3x_2^2 - 0.0022x_1x_2^4$$

$$u_{12}(x_1, x_2) = -2.05x_2 - 0.0022x_2^5 - 0.0044x_1^2x_2^3 - 0.0022x_1^4x_2$$

where insignificant components in feedback are discarded for simplicity.

In a small neighborhood about zero, decay of distortions is almost controlled by a relatively high linear feedback gain that may cause side effects for larger perturbations. To reduce such gain, we adopt in feedback a continuous but nonsmooth component that varies sharply near zero and varies slowly for larger deviations. This feedback is represented explicitly as

$$U(a) = ka^{q/p} + \cdots$$

where q and p are positive integers and the dots stand for smooth components. If $q < p$, the first term in the last formula sharply varies near zero. We compare the stabilization conditions provided by a smooth (U_1) and the following continuous and compound controllers:

$$U_2 = -0.278a^{\frac{1}{3}}$$

$$U_3 = -0.278a^{\frac{1}{3}} - 0.0350a^5 - 0.00260a^7 - 0.000210a^9$$

which satisfy a uniform normalized condition

$$G_1^0(a)\big|_{a=0.05} = G_2^0(a)\big|_{a=0.05} = G_3^0(a)\big|_{a=0.05} = -0.1$$

Figures 1a and 1b compare the stability conditions yielded by these three controllers. Figure 1a plots functions G_i^0, $i = 1, 2, 3$ vs a. A

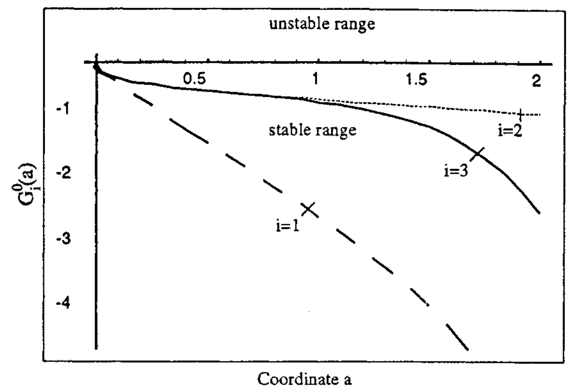

Fig. 1a Comparison of compound and smooth resonance control design approaches for the Hopf bifurcation.

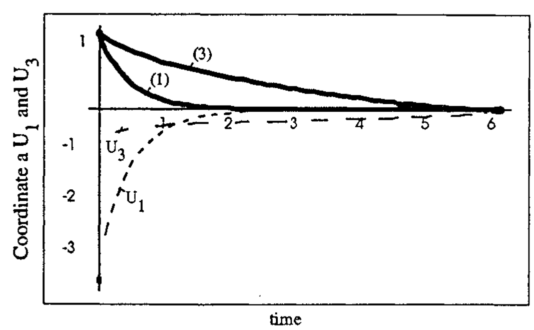

Fig. 1b Combined graph of controlled amplitudes time histories 1 and 3 provided by smooth control (U_1) and compound control (U_2).

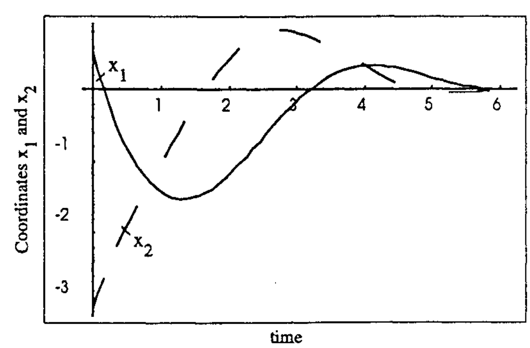

Fig. 1c Decay of phase variables of Van der Pol equation induced by compound feedback control U_3.

distance from a curve to the horizontal axis measures a local degree of stability for a chosen value of a (i.e., instantaneous degree of asymptotic convergence). Although due to the normalized condition all controllers offer similar performance near zero, both U_2 and U_3 work more efficiently for larger deviations. Figure 1b plots time histories of controlled amplitudes (thick lines) and the time evolution of controllers $U_1(a)$ and $U_3(a)$. Figure 1c plots time histories of initial coordinates x_1 and x_2 controlled by feedback $u_3(x)$, which is determined by the mapping of $U_3(a)$ into the original coordinate basis. Notice that this control is described as follows: $u_3 = \{u_{31}, u_{32}\}$, where

$$u_{31}(x_1, x_2) = -0.28x_1(0.25x_1 + 0.25x_2)^{-\frac{1}{3}} - 0.0022x_1^5 - 0.0044x_1^3x_2^2 - 0.0022x_1x_2^4$$

$$u_{32}(x_1, x_2) = -0.28x_2(0.25x_1 + 0.25x_2)^{-\frac{1}{3}} - 0.0022x_2^5 - 0.0044x_1^2x_2^3 - 0.0022x_1^4x_2$$

Analysis and Control of Bifurcation Behavior in an Aircraft Model

In this section we apply the generalized normal forms method[8] to the normal forms reduction of a close-to-critical nonlinear model of aircraft. The obtained averaging normal form reveals the failure of linearization approaches in critical cases. We begin with an analysis of the limitations of controlling a trim condition of an aircraft's unstable mode by altering both the aileron and elevator deflections about the critical values in the open-loop regime. We will show that the stability region can be enlarged with the aid of nonlinear feedback controller. The considered model was adopted from Cochran and Ho,[1] which they termed aircraft A. Governing equations of aircraft flight are written in conventional notation:

$$\dot{\alpha} = q - p\beta - \left(\frac{\rho SVC_{L\alpha}}{2m}\right)\alpha, \qquad \dot{\beta} = p\alpha - r + \left(\frac{\rho SVC_{y\beta}}{2m}\right)\beta$$

$$I_x\dot{p} = \tfrac{1}{2}\rho SV^2b\big[C_{t_\beta}\beta + C_{t_{\delta a}}\delta_a + (b/2V)\big(C_{t_p}p + C_{t_r}r\big)\big] - (I_z - I_y)qr$$

$$I_y\dot{q} = \frac{1}{2}\rho SV^2\bar{c}\left\{C_{m_\alpha}\alpha + C_{m_{\delta_e}}\delta_e + \frac{\bar{c}}{2V}\left[\left(C_{m_q} + C_{m_{\dot\alpha}}\right)q - C_{m_{\dot\alpha}}p\beta - \left(\frac{\rho SVC_{m_{\dot\alpha}}C_{L_\alpha}}{2m}\right)\alpha\right]\right\} - (I_x - I_z)pr$$

$$I_z\dot{r} = \tfrac{1}{2}\rho SV^2b\big[C_{n_\beta}\beta + (b/2V)\big(C_{n_p}p + C_{n_r}r\big)\big] - (I_y - I_x)qp$$

where α is angle of attack; p, q, and r are rolling, pitching, and yawing angular rates; and δ_a and δ_e are aileron and elevator deflections. We adopt the parameter values from Ref. 1. Note that the aircraft steady-state solution (trim condition) is a function of control parameters δ_a and δ_e. There is a transition of stability in the model linearized about the trim condition when one of its real negative eigenvalues passes through zero. This phenomenon, known as saddle-node bifurcation, is clearly described by the reduced model given by the normal forms method. Represent the preceding equations in the form

$$\dot{x}_1 = -2.9998x_1 - x_2x_3 + x_4, \qquad \dot{x}_2 = -0.05586x_2 + x_1x_3 - x_5$$
$$\dot{x}_3 = -601.37(3.8170 + \alpha_a) - 202.96x_2 - 39.970x_3 + 2.7943x_5 - 0.70574x_4x_5$$
$$\dot{x}_4 = -61.129(0.58656 + \alpha_e) - 22.692x_1 + 0.71992x_2x_3 - 4.0556x_4 + 0.95965x_3x_5 \quad (31)$$
$$\dot{x}_5 = 6.8294x_2 - 0.78674x_3x_4 - 0.47937x_5$$

where $\alpha_a = \delta_a - 3.8170$ and $\alpha_e = \delta_e - 0.58656$ are deviations of control parameters about the critical values. For the preceding system the trim condition is $x_1 = 5.65$, $x_2 = -9.79$, $x_3 = 4.77$, $x_4 = -23.9$, and $x_5 = 24.13$. Shifting the origin of coordinates to the trim condition, one is able to write Eqs. (31) in the eigenbasis of the linearized system. These cumbersome last equations are excluded from this paper. Next, we compute a first-order approximation to the normal forms of these equations and represent the normalized equations in the amplitude-phase variables as follows:

$$\dot{A}_1 = -18.915A_1 + 0.55032A_1z, \qquad \dot{\phi}_1 = -6.0434z$$
$$\dot{A}_2 = -4.8653A_2 - 0.92935A_2z, \qquad \dot{\phi}_2 = -0.056328z \quad (32)$$
$$\dot{z} = 1.5288\alpha_a - 12.407\alpha_e + 0.00073z + 0.37029z^2$$

Stability transition of the last system (which is linked with the saddle-node bifurcation) is governed by the integrable last equation of Eqs. (32). Notice that this equation is the one projected on the invariant center manifold. The center manifold equation is decoupled from the remaining stable and integrable subsystem. A comparison of the analytical solution with numerical integration of Eqs. (31) verifies that the analytical solution yields an accurate approximation, allowing us to rely upon it for design of a stabilizing feedback control.

In the case considered, the saddle-node bifurcation phenomenon is described by the quadratic equation:

$$f(z) = 0.37029z^2 + 0.00073z + 1.5288\alpha_a - 12.407\alpha_e = 0 \quad (33)$$

depending upon controllers α_a and α_e. System (32) is unstable for complex roots of Eq. (33). Real roots (say, $z_1 > z_2$) initialize an unstable saddle and a stable node correspondingly. The node attracts any trajectory departing from $z(t = 0) < z_1$. Trajectories starting from larger initial deviations are repelled by the saddle. Altering values of α_a and α_e, a pilot may adjust the stability basin but is unable to reconfigure the intrinsic bifurcation topology of the system. This can be done by a closed-loop nonlinear controller that we design using the resonance control methodology.

Computing the second iteration, we determine an upper bound on the residual for the last equation of (32). This bound is added to the resonance nonlinear terms for the critical variable z so that it may be compensated for in the control design. Let

$$G_i^0(z) = 0.00073z + 0.37029z^2 + \max[E(a^0, z)] + U_i(z)$$

where the index i is used to distinguish the control laws. In this case $\max[E(a^0, z)] = 0.69 \times 10^{-5}$ at $a^0 = 0.05$.

We now compare a few approaches in the design of resonance control by assuming that $\alpha_a = \alpha_a(z)$ and $\alpha_e = \alpha_e(z)$. We denote for simplicity a general controller

$$U(z) = 1.5288\alpha_a(z) - 12.407\alpha_e(z) \quad (34)$$

which integrates the effect of both α_a and α_e controllers.

Let us first choose the control $U(z) = U_1(z)$ as follows:

$$U_1(z) = k_1z - 0.0008z - 0.4z^2 \quad (35)$$

To satisfy Eq. (34), we assume that

$$\alpha_a = \alpha_a^1z + \alpha_a^2z^2, \qquad \alpha_e = \alpha_e^1z + \alpha_e^2z^2$$

where α_a^i and α_e^i, $i = 1, 2$, are not uniquely determined by Eq. (35).

The coefficient $k_1 = -10$ is chosen to satisfy Eq. (26) with $d = 0.05$ and $\mu = 0.005$.

Let us examine how nonsmooth components in the control law affect stabilization of the aircraft model. We again use feedback of the form $U_i = k_iz^{q/p} + \cdots$, $i = 2, 3, 4$ where $q = 1$, $p = 3$ or 5, and the dots stand for smooth components. In this case we also assume the relevant representation for α_a and α_e:

$$\alpha_a = \alpha_a^1z^{p/q} + \cdots, \qquad \alpha_e = \alpha_e^1z^{p/q} + \cdots$$

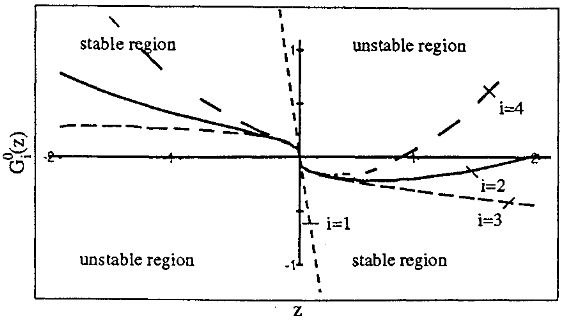

Fig. 2a Comparison of compound and smooth resonance control design approaches for model A aircraft. The vertical axis plots functions $G_i^0(z)$, and the horizontal axis plots the variable z.

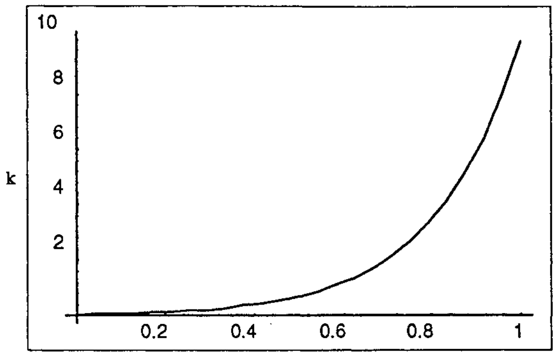

Fig. 2b Decay of linear control magnitude k due to $q/p \to 0$.

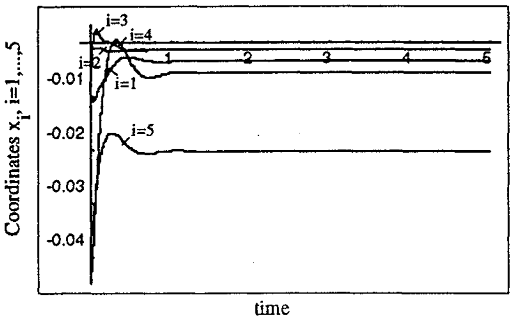

Fig. 2c Uncontrolled analytical solutions of model A aircraft.

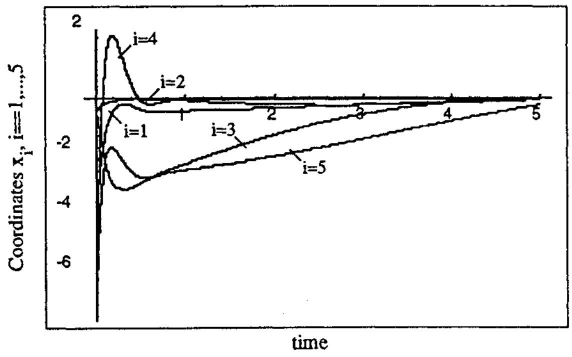

Fig. 2d Decay of model A aircraft initial variables induced by compound control U_3.

We consider the following controllers:

$$U_2(z) = -0.2925z^{\frac{1}{3}}, \qquad U_3(z) = -0.2924z^{\frac{1}{3}} - 0.0008z - 0.4z^2$$

$$U_4(z) = -0.1443z^{\frac{1}{5}}$$

where the k_i in the feedback definitions have been chosen to satisfy the uniform matching conditions:

$$G_i^0(z)\big|_{z=0.005} = -0.05, \qquad i = 2, 3, 4$$

Comparison of the designed controllers is shown on Fig. 2a plotting functions $G_i^0(z), i = 1, \ldots, 4$ (note that the small-dashed, solid, medium-dashed, and larger dashed lines correspond to $i = 1, \ldots, 4$, respectively). Notice that a controller asymptotically stabilizes the plant for such values of z where the related curve is located either in the second or in the fourth quadrants (see Fig. 2a). For example, the stabilization provided by controllers U_2 and U_4 violates this condition when the related curve crosses the z axis. Inside of the stability region, a distance from a curve to the z axis measures a degree of local asymptotic stability (i.e., measures the instantaneous rate of perturbation decay with the chosen initial magnitude). The controllers show close performance for small deviations; however, the performance for larger ones turns out to be distinct. It is clear that a designer is able to better match robustness and stability requirements by engaging continuous but nonsmooth components in feedback. Figure 2b shows decay of control magnitude for controller U_3 when $q/p \to 0$. Figure 2c plots time histories of uncontrolled initial variables, and Fig. 2d plots decay of initial plant variables induced by control U_3. As is clear from Fig. 2a, the compound control (U_3) better matches efficiency and stability requirements.

Conclusions

High speed and maneuverability are the principal expectations in design of a modern aircraft. Aircraft become more maneuverable near extrema of the flight envelope, where it is usually more difficult to control. Stabilization by a linear control in an extended neighborhood about the operating regime often enlarges feedback gains that degrade maneuverability. In this paper we have presented a new approach to designing compound robust resonance control that efficiently stabilizes aircraft transition behavior in a certain region about the operating conditions. Our feedback consists of continuous (nonsmooth) and conventional smooth nonlinear components. The nonsmooth component contributes primarily in a small region near zero, whereas smooth nonlinear components take over for the larger deviations from the trim conditions. The proposed approach adopts profound features of discontinuous and smooth design procedures and escapes their side effects.

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